

The linear stability of the post-Newtonian triangular equilibrium in the three-body problem

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(Dated: December 28, 2016)

Abstract

Continuing work initiated in an earlier publication [Yamada, Tsuchiya, and Asada, Phys. Rev. D **91**, 124016 (2015)], we reexamine the linear stability of the triangular solution in the relativistic three-body problem for general masses by the standard linear algebraic analysis. In this paper, we start with the Einstein-Infeld-Hoffman form of equations of motion for N -body systems in the uniformly rotating frame. As an extension of the previous work, we consider general perturbations to the equilibrium, i.e. we take account of perturbations orthogonal to the orbital plane, as well as perturbations lying on it.

It is found that while the orthogonal perturbations are independent of the lying ones likewise the Newtonian case, these depend on each other by the 1PN three-body interactions. We also show that the orthogonal perturbations do not affect the condition of stability. This is because these always precess with two frequency modes; the same with the orbital frequency and the slightly different one by the 1PN effect. The same condition of stability with the previous one, which is valid even for the general perturbations, is obtained from the lying perturbations.

PACS numbers: 04.25.Nx, 45.50.Pk, 95.10.Ce, 95.30.Sf

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I. INTRODUCTION

The direct detections of gravitational waves from merger of binary black hole by Advanced LIGO have opened a new window to test general relativity [1–3]. In the near future, gravitational waves astronomy will be largely developed by a network of gravitational wave detectors such as Advanced VIRGO [4] and KAGRA [5]. One of the most promising astrophysical sources is inspiraling and merging binary compact stars. In fact, the two events of Advanced LIGO fit well with binary black hole mergers [1–3].

With growing interest, gravitational waves involving three-body interactions have been discussed (e.g., [6–10]). Even in Newtonian gravity, the three-body problem is not integrable by analytical methods. As particular solutions, however, Euler and Lagrange found a collinear solution and an equilateral triangular one, respectively. The solutions to the restricted three-body problem, where one of the three bodies is a test particle, are known as Lagrangian points [11–13]. In particular, Lagrange’s equilateral triangular orbit has stimulated renewed interest for relativistic astrophysics [14–22]. Recently, a relativistic hierarchical triple system has been discovered for the first time [23], and dynamics of such systems has also been studied by several authors [24–29].

For three finite masses, in the first post-Newtonian (1PN) approximation, the existence and uniqueness of a post-Newtonian (PN) collinear solution corresponding to Euler’s one have been shown by Yamada and Asada [30, 31]. Also, Ichita *et al.*, including one of the present authors, have shown that an equilateral triangular solution is possible at the 1PN order, if and only if all the three masses are equal [18]. Generalizing this earlier work, Yamada and Asada have found a *PN triangular* equilibrium solution for general masses with 1PN corrections to each side length [19]. This PN triangular configuration for general masses is not always equilateral and it recovers the previous results for the restricted three-body case [32, 33].

In Newtonian gravity, Gascheau proved that Lagrange’s equilateral triangular configuration for circular motion is stable [34], if

$$\frac{m_1 m_2 + m_2 m_3 + m_3 m_1}{M^2} < \frac{1}{27}, \quad (1)$$

where M is the total mass. Routh extended the result to a general law of gravitation $\propto 1/r^k$,

and found the condition for stability as [35]

$$\frac{m_1 m_2 + m_2 m_3 + m_3 m_1}{M^2} < \frac{1}{3} \left(\frac{3-k}{1+k} \right)^2. \quad (2)$$

The condition of stability (1) has recently been corrected in the 1PN approximation as [20]

$$\frac{m_1 m_2 + m_2 m_3 + m_3 m_1}{M^2} + \frac{15}{2} \frac{m_1 m_2 m_3}{M^3} \varepsilon < \frac{1}{27} \left(1 - \frac{391}{54} \varepsilon \right), \quad (3)$$

where we define

$$\varepsilon \equiv \left(\frac{GM\omega}{c^3} \right)^{2/3}, \quad (4)$$

with the common orbital frequency ω of the system. To derive the condition (3), only the perturbations in the orbital plane are taken into account in the previous paper [20]. In Newtonian gravity, it is reasonable because the perturbations orthogonal to the orbital plane always oscillate with the orbital frequency. However, it is not obvious whether this is the case at the 1PN order.

Therefore, the main purpose of the present paper is to take account of the perturbations orthogonal to the orbital plane, as well as those lying on it, in order to show that the perturbations orthogonal to the orbital plane always oscillate even at the 1PN order and they do not affect the condition of stability. We also derive the condition of stability from the motion of lying perturbations by the standard linear algebraic analysis.

This paper is organized as follows. In Sec. II, we briefly summarize the PN triangular equilibrium solution for three finite masses in the corotating frame. The Perturbations orthogonal to the orbital plane are discussed in Sec. III. In Sec. IV, we consider the perturbations lying on the orbital plane in order to derive the condition of stability. Section V is devoted to the conclusion.

II. THE PN TRIANGULAR EQUILIBRIUM SOLUTION IN THE COROTATING FRAME

Following Ref. [19], we summarize a derivation of PN triangular equilibrium solution for general masses in this section. In order to take account of the terms at the 1PN order, we employ the Einstein-Infeld-Hoffman (EIH) form of the equations of motion for N -body

systems in uniformly rotating frame (please see Appendix A for the derivation):

$$\begin{aligned}
\frac{d^2 \mathbf{r}_K}{dt^2} = & \sum_{A \neq K} \frac{Gm_A}{r_{KA}^3} \mathbf{r}_{AK} - 2(\boldsymbol{\Omega} \times \mathbf{v}_K) - (\boldsymbol{\Omega} \cdot \mathbf{r}_K) \boldsymbol{\Omega} + \Omega^2 \mathbf{r}_K \\
& + \sum_{A \neq K} \frac{Gm_A}{r_{KA}^3} \mathbf{r}_{AK} \left[-4 \sum_{B \neq K} \frac{Gm_B}{c^2 r_{KB}} - \sum_{C \neq A} \frac{Gm_C}{c^2 r_{AC}} \left(1 + \frac{\mathbf{r}_{AK} \cdot \mathbf{r}_{AC}}{2r_{CA}^2} \right) \right. \\
& + \left(\frac{\mathbf{v}_K + (\boldsymbol{\Omega} \times \mathbf{r}_K)}{c} \right)^2 + 2 \left(\frac{\mathbf{v}_A + (\boldsymbol{\Omega} \times \mathbf{r}_A)}{c} \right)^2 \\
& - 4 \left(\frac{\mathbf{v}_K + (\boldsymbol{\Omega} \times \mathbf{r}_K)}{c} \right) \cdot \left(\frac{\mathbf{v}_A + (\boldsymbol{\Omega} \times \mathbf{r}_A)}{c} \right) - \frac{3}{2} \left\{ \left(\frac{\mathbf{v}_A + (\boldsymbol{\Omega} \times \mathbf{r}_A)}{c} \right) \cdot \mathbf{x}_{AK} \right\}^2 \Big] \\
& - \sum_{A \neq K} \frac{Gm_A}{c^2 r_{KA}^2} \left[\mathbf{x}_{AK} \cdot \left(\frac{4[\mathbf{v}_K + (\boldsymbol{\Omega} \times \mathbf{r}_K)] - 3[\mathbf{v}_A + (\boldsymbol{\Omega} \times \mathbf{r}_A)]}{c} \right) \right] \\
& \times \left(\frac{[\mathbf{v}_K + (\boldsymbol{\Omega} \times \mathbf{r}_K)] - [\mathbf{v}_A + (\boldsymbol{\Omega} \times \mathbf{r}_A)]}{c} \right) \\
& + \frac{7}{2} \sum_{A \neq K} \sum_{C \neq A} \frac{Gm_A}{c^2 r_{KA}} \frac{Gm_C}{r_{AC}^3} \mathbf{r}_{CA}, \tag{5}
\end{aligned}$$

where $\mathbf{A} \times \mathbf{B}$ and $\mathbf{A} \cdot \mathbf{B}$ denote the outer product and the inner product of vectors \mathbf{A} and \mathbf{B} in the Euclidian space, \mathbf{r}_K and \mathbf{v}_K are the position and velocity of each body in the rotating frame, respectively, $\boldsymbol{\Omega}$ is a uniform angular velocity of the coordinate respect to an inertial frame, and we define

$$\Omega \equiv |\boldsymbol{\Omega}|, \tag{6}$$

$$\mathbf{r}_{AK} \equiv \mathbf{r}_A - \mathbf{r}_K, \tag{7}$$

$$r_{AK} \equiv |\mathbf{r}_{AK}|, \tag{8}$$

$$\mathbf{x}_{AK} \equiv \frac{\mathbf{r}_{AK}}{r_{AK}}. \tag{9}$$

In the following, we assume circular motion of bodies.

Let us consider a PN triangular configuration with 1PN corrections to each side length of a Newtonian equilateral triangle, so that the distances between the bodies are expressed

$$r_{IJ} = \ell(1 + \rho_{IJ}), \tag{10}$$

where $I, J = 1, 2, 3$ and $\rho_{IJ}(= \rho_{JI})$ is dimensionless PN corrections (see Fig. 1). Because of circular motion, ℓ and ρ_{IJ} are constants. Note that we neglect the terms of second (and higher) order in ε henceforth. Here, if all the three corrections are equal (i.e. $\rho_{12} = \rho_{23} = \rho_{31} = \rho$), a PN configuration is still an equilateral triangle, though each side length is

changed by a scale transformation as $\ell \rightarrow \ell(1 + \rho)$. Namely, one of the degrees of freedom for the PN corrections corresponds to a scale transformation, and this is unimportant. In order to eliminate this degree of freedom, we impose a constraint condition

$$\frac{r_{12} + r_{23} + r_{31}}{3} = \ell, \quad (11)$$

which means that the arithmetical mean of the three distances of the bodies is not changed by the PN corrections. Namely,

$$\rho_{12} + \rho_{23} + \rho_{31} = 0. \quad (12)$$

Please see also Ref. [19] for imposing this constraint.

The PN triangular solution for general masses is a coplanar equilibrium, in which three bodies rest in Eq. (5), therefore

$$\frac{d^2 \mathbf{r}_K}{dt^2} = \mathbf{v}_K = \mathbf{0}, \quad (13)$$

$$\boldsymbol{\Omega} \cdot \mathbf{r}_K = 0, \quad (14)$$

where we take the origin of the coordinate as the center of mass. Straightforward calculations lead to

$$\rho_{12} = \frac{1}{24}[(\nu_2 - \nu_3)(5 - 3\nu_1) - (\nu_3 - \nu_1)(5 - 3\nu_2)]\varepsilon, \quad (15)$$

$$\rho_{23} = \frac{1}{24}[(\nu_3 - \nu_1)(5 - 3\nu_2) - (\nu_1 - \nu_2)(5 - 3\nu_3)]\varepsilon, \quad (16)$$

$$\rho_{31} = \frac{1}{24}[(\nu_1 - \nu_2)(5 - 3\nu_3) - (\nu_2 - \nu_3)(5 - 3\nu_1)]\varepsilon, \quad (17)$$

with $\nu_I \equiv m_I/M$. In this case, the common angular velocity is given by

$$\Omega = \omega = \omega_N(1 + \tilde{\omega}_{\text{PN}}), \quad (18)$$

where

$$\omega_N \equiv \sqrt{\frac{GM}{\ell^3}}, \quad (19)$$

$$\tilde{\omega}_{\text{PN}} \equiv -\frac{1}{16}(29 - 14V)\varepsilon, \quad (20)$$

with $V \equiv \nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1$. $V = 0$ means two of the three masses are zero, thus we consider the case of $V \neq 0$ in this paper. Hereafter, we take the units of $G = c = 1$.

Before closing this section, let us denote perturbations to the positions as

$$\mathbf{r}_I \rightarrow \mathbf{r}_I + \delta\mathbf{r}_I, \quad (21)$$

and define the relative perturbations as

$$\delta\mathbf{r}_{IJ} \equiv \delta\mathbf{r}_I - \delta\mathbf{r}_J, \quad (22)$$

with

$$\delta\mathbf{r}_{IJ} = r_{IJ} (\xi_{IJ}\mathbf{x}_{IJ} + \eta_{IJ}\mathbf{y}_{IJ} + \zeta_{IJ}\mathbf{z}), \quad (23)$$

where $\mathbf{z} \equiv \mathbf{\Omega}/\Omega$ and $\mathbf{y}_{IJ} \equiv \mathbf{z} \times \mathbf{x}_{IJ}$. Obviously, we can obtain

$$|\mathbf{r}_{IJ}| \rightarrow |\mathbf{r}_{IJ} + \delta\mathbf{r}_{IJ}| = r_{IJ}(1 + \xi_{IJ}), \quad (24)$$

at the 1PN order. By the definition of $\delta\mathbf{r}_{IJ}$ (22), we obtain

$$\delta\mathbf{r}_{12} + \delta\mathbf{r}_{23} + \delta\mathbf{r}_{31} = \mathbf{0}. \quad (25)$$

III. ORTHOGONAL PERTURBATIONS

In this section, we focus on the perturbations orthogonal to the orbital plane. The z -direction of perturbed equations of motion is expressed as

$$\begin{aligned} \ddot{\delta\mathbf{r}}_I \cdot \mathbf{z} = & -\frac{m_J}{\ell^2}\zeta_{IJ} + \frac{m_K}{\ell^2}\zeta_{KI} + \varepsilon\frac{M}{\ell^2} \left[\frac{1}{24}\nu_J [36\nu_J^2 + 36\nu_J(\nu_K - 1) + 45\nu_K^2 - 18\nu_K + 82] \zeta_{IJ} \right. \\ & \left. - \frac{1}{24}\nu_K [45\nu_J^2 + 18\nu_J(2\nu_K - 1) + 36\nu_K^2 - 36\nu_K + 82] \zeta_{KI} - \frac{\sqrt{3}}{2}\nu_J\nu_K \left(\frac{\dot{\zeta}_{IJ} + \dot{\zeta}_{KI}}{\omega_N} \right) \right], \end{aligned} \quad (26)$$

where the dot denotes the derivatives with respect to time. Immediately, one can find that the orthogonal perturbations are independent of the lying ones. Therefore, we can study whether a conditions of stability for z -direction exists or not, separately from the condition for ξ_{IJ} and η_{IJ} .

In fact, one can infer from Eq. (5) that the z -direction of perturbed equations of motion is separated from ξ_{IJ} and η_{IJ} . This is because the contributions to the z -direction must come from the terms parallel to $\mathbf{\Omega}$ except for $\delta\mathbf{r}$ itself. Therefore, if ξ_{IJ} and η_{IJ} contribute

to the z -direction, the terms of the form $(\delta \mathbf{r} \cdot \mathbf{r})\mathbf{\Omega}$ must appear. However, in Eq. (5), the terms parallel to $\mathbf{\Omega}$ is only of the form $(\mathbf{\Omega} \cdot \mathbf{r})\mathbf{\Omega}$.

Moreover, we can separate motion of the common center of mass from that of the relative perturbations. The z -direction of equations of motion for the common center of mass is straightforwardly calculated from Eq. (26), and this becomes

$$\sum_{I=1}^3 m_I \delta \ddot{\mathbf{r}}_I \cdot \mathbf{z} = 0. \quad (27)$$

Note that although the position of the PN center of mass is different from that of the Newtonian one in general, we can use the same expressions for the orthogonal perturbations in the PN triangular equilibrium. Therefore, the common center of mass is always in uniform linear motion for z -direction, and then, this do not affect the condition of stability.

From Eq. (25), we can find a relation for the relative perturbations of z -direction as

$$\zeta_{12} + \zeta_{23} + \zeta_{31} = 0. \quad (28)$$

Therefore, the degrees of freedom of ζ_{IJ} is two. Let us eliminate ζ_{23} by using this relation, so that the perturbed equations of motion for z -direction can be expressed as

$$D\boldsymbol{\zeta} = \mathcal{M}\boldsymbol{\zeta},$$

$$\mathcal{M} \equiv \begin{pmatrix} -1 + \varepsilon A & \frac{1}{2}\nu_3(\nu_1 - \nu_2)\varepsilon & -\frac{\sqrt{3}}{2}\nu_2\nu_3\varepsilon & \frac{\sqrt{3}}{2}\nu_3(\nu_1 + \nu_2)\varepsilon \\ -\frac{1}{2}\nu_2(\nu_3 - \nu_1)\varepsilon & -1 + \varepsilon B & -\frac{\sqrt{3}}{2}\nu_2(\nu_3 + \nu_1)\varepsilon & \frac{\sqrt{3}}{2}\nu_2\nu_3\varepsilon \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (29)$$

where $\boldsymbol{\zeta} \equiv (D\zeta_{12}, D\zeta_{31}, \zeta_{12}, \zeta_{31})$ with $D \equiv d/\omega_N dt$ and

$$A \equiv \frac{1}{8} (6\nu_1^2 + 2\nu_1\nu_2 - 6\nu_1 + 10\nu_2^2 - 10\nu_2 + 29),$$

$$B \equiv \frac{1}{8} (6\nu_1^2 + 2\nu_1\nu_3 - 6\nu_1 + 10\nu_3^2 - 10\nu_3 + 29). \quad (30)$$

The eigenvalues of the matrix \mathcal{M} are

$$\lambda_1 = -i(1 + \tilde{\omega}_{\text{PN}}), \quad \lambda_2 = i(1 + \tilde{\omega}_{\text{PN}}), \quad \lambda_3 = -i(1 + \tilde{\omega}_{\text{X}}), \quad \lambda_4 = i(1 + \tilde{\omega}_{\text{X}}), \quad (31)$$

where

$$\tilde{\omega}_{\text{X}} \equiv -\frac{1}{16} (29 - 6V) \varepsilon. \quad (32)$$

Since $\varepsilon \ll 1$, each λ is always purely imaginary. Note that neglecting the higher order in ε for the eigenequation leads the incorrect eigenvalues, because the eigenvalues are degenerate in the Newtonian limit. Therefore, we neglect the higher order after solving the eigenequation.

Since all the eigenvalues are different from each other, there is a regular matrix P such that $P^{-1}\mathcal{M}P = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Therefore, Eq. (29) can be rewritten as

$$D\hat{\zeta} = P^{-1}\mathcal{M}P\hat{\zeta}, \quad (33)$$

where $\hat{\zeta} \equiv P^{-1}\zeta$. We can solve the above equation for $\hat{\zeta}$ as

$$\hat{\zeta} = \exp(\omega_{\text{N}}tP^{-1}\mathcal{M}P)\hat{\zeta}_0 = \text{diag}(e^{\lambda_1\omega_{\text{N}}t}, e^{\lambda_2\omega_{\text{N}}t}, e^{\lambda_3\omega_{\text{N}}t}, e^{\lambda_4\omega_{\text{N}}t})\hat{\zeta}_0, \quad (34)$$

where $\hat{\zeta}_0$ is the initial value. Thus, the perturbations ζ can be expressed by using the trigonometric functions, and hence ζ_{IJ} always oscillate with two frequency modes; $(1 + \tilde{\omega}_{\text{PN}})$ and $(1 + \tilde{\omega}_{\text{X}})$. Therefore, the PN triangular equilibrium is stable for the perturbations orthogonal to the orbital plane likewise the Newtonian case. It is worthwhile to mention that in contrast to the Newtonian case ζ_{IJ} has the mode $(1 + \tilde{\omega}_{\text{X}})$, which is different from the orbital frequency. This might occur *resonant orbits* in the nonlinear analysis [36].

IV. LYING PERTURBATIONS

Next, we consider the perturbations in the orbital plane. For these perturbations, the motion of the center of mass is slightly complicated. This is because the PN corrections to the position of the center of mass can not be canceled for these perturbations. However, the motion of the center of mass is not important to consider the stability of the PN triangular configuration. Hence, we focus on the relative perturbations ξ_{IJ} and η_{IJ} in this paper.

By separating the motion for the center of mass, it is sufficient to discuss the remaining four degrees of freedom for ξ_{IJ} and η_{IJ} . For the relative perturbations in the orbital plane, it is convenient to use Routh's variables χ_{12} , X , ψ_{23} , and σ , in which we consider the relative perturbations to \mathbf{r}_1 and \mathbf{r}_3 with fixed \mathbf{r}_2 . Here, χ_{12} and σ correspond to the scale transformation of the triangle and the change of the angle of the system to a reference direction, respectively. On the other hand, X and ψ_{23} are the degrees of freedom of a shape change from the equilateral triangle. Figure 2 shows the perturbations using Routh's

variables. By the linear transformations as (please see also Appendix B)

$$\chi_{12} = (1 + \rho_{12})\xi_{12}, \quad (35)$$

$$X = (1 + \rho_{31})\xi_{31} - (1 + \rho_{12})\xi_{12}, \quad (36)$$

$$\psi_{23} = \eta_{31} - \eta_{12}, \quad (37)$$

$$\sigma = \eta_{12}, \quad (38)$$

we obtain the equations of motion as

$$\begin{aligned} 0 = & (D^2 - 3) \chi_{12} - 2D\sigma - \frac{9}{4}\nu_3 X - \frac{3}{4}\sqrt{3}\nu_3\psi_{23} + \varepsilon \left[\left(\frac{1}{32} [-11\nu_2^2(9\nu_3 + 8) + \nu_2(-72\nu_3^2 \right. \right. \\ & - 34\nu_3 + 88) + 63\nu_3^3 - 34\nu_3^2 + 16\nu_3 + 540] - \frac{1}{8}\sqrt{3}D\nu_3(9\nu_3 - 7)(2\nu_2 + \nu_3 - 1) \Big) \chi_{12} \\ & + \frac{1}{24}D [-6\nu_2^2(9\nu_3 + 19) - 6\nu_2(9\nu_3^2 + 10\nu_3 - 19) + 27\nu_3^3 - 60\nu_3^2 + 63\nu_3 + 125] \sigma \\ & + \left(\frac{1}{32}\nu_3 [99\nu_2^2 + 2\nu_2(27\nu_3 - 85) + 171\nu_3^2 - 304\nu_3 + 553] - \frac{1}{8}\sqrt{3}D\nu_3 [\nu_2(9\nu_3 + 7) \right. \\ & + 9\nu_3^2 - 12\nu_3 - 1] \Big) X + \left(\frac{1}{8}D\nu_3 [-\nu_2(9\nu_3 + 7) + 9\nu_3^2 - 32\nu_3 + 11] + \frac{1}{32}\sqrt{3}\nu_3 [24\nu_2^2 \right. \\ & + \nu_2(60\nu_3 + 62) + 87\nu_3^2 - 54\nu_3 + 122] \Big) \psi_{23} \Big], \quad (39) \end{aligned}$$

$$\begin{aligned} 0 = & D^2\sigma + 2D\chi_{12} - \frac{3}{4}\sqrt{3}\nu_3 X + \frac{9}{4}\nu_3\psi_{23} + \varepsilon \left[\left(\frac{1}{8}D [-6\nu_2^2(3\nu_3 + 5) - 6\nu_2(3\nu_3^2 + 2\nu_3 - 5) \right. \right. \\ & + 9\nu_3^3 - 36\nu_3^2 + 27\nu_3 - 61] + \frac{3}{32}\sqrt{3}\nu_3 [-3\nu_2^2 + \nu_2(24\nu_3 - 34) + 15\nu_3^2 - 26\nu_3 + 16] \Big) \chi_{12} \\ & + \left(\frac{1}{24}D^2 [6\nu_2^2 + 6\nu_2(\nu_3 - 1) - 3\nu_3^2 - 12\nu_3 + 5] + \frac{1}{8}\sqrt{3}D\nu_3(9\nu_3 - 13)(2\nu_2 + \nu_3 - 1) \Big) \sigma \\ & + \left(\frac{1}{8}D\nu_3 (-9\nu_2\nu_3 + \nu_2 + 9\nu_3^2 - 34\nu_3 + 13) + \frac{1}{32}\sqrt{3}\nu_3 [45\nu_2^2 + 18\nu_2(5\nu_3 - 3) + 81\nu_3^2 \right. \\ & - 72\nu_3 + 151] \Big) X + \left(\frac{1}{8}\sqrt{3}D\nu_3 [\nu_2(9\nu_3 - 1) + 9\nu_3^2 - 18\nu_3 + 5] - \frac{9}{32}\nu_3 [20\nu_2^2 \right. \\ & + 2\nu_2(6\nu_3 - 7) + 13\nu_3^2 - 18\nu_3 + 50] \Big) \psi_{23} \Big], \quad (40) \end{aligned}$$

$$\begin{aligned}
0 = & (D^2 - 3) \chi_{12} - 2D\sigma + \left(D^2 + \frac{9\nu_2}{4} - 3 \right) X + \left(-2D - \frac{3\sqrt{3}\nu_2}{4} \right) \psi_{23} \\
& + \varepsilon \left[\left(\frac{1}{8} \sqrt{3} D \nu_2 (9\nu_2 - 7)(\nu_2 + 2\nu_3 - 1) \right. \right. \\
& + \frac{1}{32} [63\nu_2^3 - 2\nu_2^2(36\nu_3 + 17) + \nu_2(-99\nu_3^2 - 34\nu_3 + 16) - 88\nu_3^2 + 88\nu_3 + 540] \Big) \chi_{12} \\
& + \frac{1}{24} D [27\nu_2^3 - 6\nu_2^2(9\nu_3 + 10) + \nu_2(-54\nu_3^2 - 60\nu_3 + 63) - 114\nu_3^2 + 114\nu_3 + 125] \sigma \\
& + \left(\frac{1}{8} \sqrt{3} D \nu_2 [\nu_2(9\nu_3 - 4) - 21\nu_3 + 8] \right. \\
& + \frac{1}{32} [-108\nu_2^3 - 18\nu_2^2(7\nu_3 - 15) + \nu_2(-198\nu_3^2 + 136\nu_3 - 537) - 88\nu_3^2 + 88\nu_3 + 540] \Big) X \\
& + \left(\frac{1}{24} D [-9\nu_2^2(3\nu_3 - 4) + \nu_2(-54\nu_3^2 - 39\nu_3 + 30) - 114\nu_3^2 + 114\nu_3 + 125] \right. \\
& \left. \left. + \frac{1}{32} \sqrt{3} \nu_2 [87\nu_2^2 + 6\nu_2(10\nu_3 - 9) + 24\nu_3^2 + 62\nu_3 + 122] \right) \psi_{23} \right], \tag{41}
\end{aligned}$$

$$\begin{aligned}
0 = & 2D\chi_{12} + D^2\sigma + \left(2D - \frac{3\sqrt{3}\nu_2}{4} \right) X + \left(D^2 - \frac{9\nu_2}{4} \right) \psi_{23} \\
& + \varepsilon \left[\left(\frac{1}{8} D [9\nu_2^3 - 18\nu_2^2(\nu_3 + 2) - 3\nu_2(6\nu_3^2 + 4\nu_3 - 9) - 30\nu_3^2 + 30\nu_3 - 61] \right. \right. \\
& - \frac{3}{32} \sqrt{3} \nu_2 [15\nu_2^2 + \nu_2(24\nu_3 - 26) - 3\nu_3^2 - 34\nu_3 + 16] \Big) \chi_{12} \\
& + \left(\frac{1}{24} D^2 [\nu_1(3\nu_2 - 6\nu_3 + 5) + \nu_2(3\nu_3 - 10) + 5\nu_3] - \frac{1}{8} \sqrt{3} D \nu_2 (9\nu_2 - 13)(\nu_2 + 2\nu_3 - 1) \right) \sigma \\
& + \left(\frac{1}{8} D [-(9\nu_3 + 2)\nu_2^2 + \nu_2(-18\nu_3^2 - 13\nu_3 + 14) - 30\nu_3^2 + 30\nu_3 - 61] \right. \\
& + \frac{1}{32} \sqrt{3} \nu_2 [36\nu_2^2 + 6\nu_2(3\nu_3 + 1) + 54\nu_3^2 + 48\nu_3 + 103] \Big) X \\
& + \left(\frac{1}{24} D^2 [\nu_1(3\nu_2 - 6\nu_3 + 5) + \nu_2(3\nu_3 - 10) + 5\nu_3] + \frac{1}{8} \sqrt{3} D \nu_2 [\nu_2(4 - 9\nu_3) + 25\nu_3 - 8] \right. \\
& \left. \left. + \frac{9}{32} \nu_2 [13\nu_2^2 + 6\nu_2(2\nu_3 - 3) + 20\nu_3^2 - 14\nu_3 + 50] \right) \psi_{23} \right]. \tag{42}
\end{aligned}$$

These are equivalent to Eqs. (25)-(28) in Ref. [20]. Note that the above equations do not contain σ . This is consistent with the fact that the initial value of σ can be zero through the appropriate coordinate rotation.

The above equations can be rewritten as

$$D\chi = \mathcal{N}\chi, \tag{43}$$

where $\chi = (D\chi_{12}, DX, D\psi_{23}, D\sigma, \chi_{12}, X, \psi_{23})$ and the components of the coefficient matrix

\mathcal{N} are

$$\begin{aligned}
\mathcal{N}_{11} &= \frac{\varepsilon}{8}\sqrt{3}\nu_3(9\nu_3 - 7)(2\nu_2 + \nu_3 - 1), \\
\mathcal{N}_{12} &= \frac{\varepsilon}{8}\sqrt{3}\nu_3[\nu_2(9\nu_3 + 7) + 9\nu_3^2 - 12\nu_3 - 1], \\
\mathcal{N}_{13} &= -\frac{\varepsilon}{8}\nu_3[-\nu_2(9\nu_3 + 7) + 9\nu_3^2 - 32\nu_3 + 11], \\
\mathcal{N}_{14} &= 2 - \frac{\varepsilon}{24}[-6\nu_2^2(9\nu_3 + 19) - 6\nu_2(9\nu_3^2 + 10\nu_3 - 19) + 27\nu_3^3 - 60\nu_3^2 + 63\nu_3 + 125], \\
\mathcal{N}_{15} &= 3 - \frac{\varepsilon}{32}[-11\nu_2^2(9\nu_3 + 8) + \nu_2(-72\nu_3^2 - 34\nu_3 + 88) + 63\nu_3^3 - 34\nu_3^2 + 16\nu_3 + 540], \\
\mathcal{N}_{16} &= \frac{9}{4}\nu_3 - \frac{\varepsilon}{32}\nu_3[99\nu_2^2 + 2\nu_2(27\nu_3 - 85) + 171\nu_3^2 - 304\nu_3 + 553], \\
\mathcal{N}_{17} &= \frac{3}{4}\sqrt{3}\nu_3 - \frac{\varepsilon}{32}\sqrt{3}\nu_3[24\nu_2^2 + \nu_2(60\nu_3 + 62) + 87\nu_3^2 - 54\nu_3 + 122], \\
\mathcal{N}_{21} &= -\frac{\varepsilon}{8}\sqrt{3}\nu_2(9\nu_2 - 7)(\nu_2 + 2\nu_3 - 1) - \frac{\varepsilon}{8}\sqrt{3}\nu_3(9\nu_3 - 7)(2\nu_2 + \nu_3 - 1), \\
\mathcal{N}_{22} &= -\frac{\varepsilon}{8}\sqrt{3}\nu_2[\nu_2(9\nu_3 - 4) - 21\nu_3 + 8] - \frac{\varepsilon}{8}\sqrt{3}\nu_3[\nu_2(9\nu_3 + 7) + 9\nu_3^2 - 12\nu_3 - 1], \\
\mathcal{N}_{23} &= 2 - \frac{\varepsilon}{24}[-9\nu_2^2(3\nu_3 - 4) + 3\nu_2(-18\nu_3^2 - 13\nu_3 + 10) - 114\nu_3^2 + 114\nu_3 + 125] \\
&\quad + \frac{\varepsilon}{8}\nu_3[-\nu_2(9\nu_3 + 7) + 9\nu_3^2 - 32\nu_3 + 11], \\
\mathcal{N}_{24} &= -\frac{\varepsilon}{24}[27\nu_2^3 - 6\nu_2^2(9\nu_3 + 10) + 3\nu_2(-18\nu_3^2 - 20\nu_3 + 21) - 114\nu_3^2 + 114\nu_3 + 125] \\
&\quad + \frac{\varepsilon}{24}[-6\nu_2^2(9\nu_3 + 19) - 6\nu_2(9\nu_3^2 + 10\nu_3 - 19) + 27\nu_3^3 - 60\nu_3^2 + 63\nu_3 + 125], \\
\mathcal{N}_{25} &= -\frac{\varepsilon}{32}[63\nu_2^3 - 2\nu_2^2(36\nu_3 + 17) + \nu_2(-99\nu_3^2 - 34\nu_3 + 16) - 88\nu_3^2 + 88\nu_3 + 540] \\
&\quad + \frac{\varepsilon}{32}[-11\nu_2^2(9\nu_3 + 8) + \nu_2(-72\nu_3^2 - 34\nu_3 + 88) + 63\nu_3^3 - 34\nu_3^2 + 16\nu_3 + 540], \\
\mathcal{N}_{26} &= -\frac{9\nu_2}{4} + 3 - \frac{\varepsilon}{32}[-108\nu_2^3 - 18\nu_2^2(7\nu_3 - 15) + \nu_2(-198\nu_3^2 + 136\nu_3 - 537) \\
&\quad - 88\nu_3^2 + 88\nu_3 + 540] - \frac{9}{4}\nu_3 + \frac{\varepsilon}{32}\nu_3[99\nu_2^2 + 2\nu_2(27\nu_3 - 85) + 171\nu_3^2 - 304\nu_3 + 553], \\
\mathcal{N}_{27} &= \frac{3\sqrt{3}\nu_2}{4} - \frac{\varepsilon}{32}\sqrt{3}\nu_2[87\nu_2^2 + 6\nu_2(10\nu_3 - 9) + 24\nu_3^2 + 62\nu_3 + 122] - \frac{3}{4}\sqrt{3}\nu_3 \\
&\quad + \frac{\varepsilon}{32}\sqrt{3}\nu_3[24\nu_2^2 + \nu_2(60\nu_3 + 62) + 87\nu_3^2 - 54\nu_3 + 122], \\
\mathcal{N}_{31} &= -\frac{\varepsilon}{24}[27\nu_2^3 - 54\nu_2^2(\nu_3 + 2) - \nu_2(54\nu_3^2 + 42\nu_3 - 101) - 90\nu_3^2 + 80\nu_3 - 183 \\
&\quad - 2\nu_1(3\nu_2 - 6\nu_3 + 5)] \\
&\quad + \frac{\varepsilon}{24}[-6\nu_2^2(9\nu_3 + 17) - 6\nu_2(9\nu_3^2 + 8\nu_3 - 17) + 27\nu_3^3 - 102\nu_3^2 + 105\nu_3 - 193],
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}_{32} &= 2 + -\frac{\varepsilon}{24}[-3(9\nu_3 + 2)\nu_2^2 - \nu_2(54\nu_3^2 + 45\nu_3 - 62) - 90\nu_3^2 + 80\nu_3 - 183 \\
&\quad - 2\nu_1(3\nu_2 - 6\nu_3 + 5)] \\
&\quad + \frac{\varepsilon}{8}\nu_3(-9\nu_2\nu_3 + \nu_2 + 9\nu_3^2 - 34\nu_3 + 13), \\
\mathcal{N}_{33} &= -\frac{\varepsilon}{8}\sqrt{3}\nu_2[\nu_2(4 - 9\nu_3) + 25\nu_3 - 8] + \frac{\varepsilon}{8}\sqrt{3}\nu_3[\nu_2(9\nu_3 - 1) + 9\nu_3^2 - 18\nu_3 + 5], \\
\mathcal{N}_{34} &= \frac{\varepsilon}{8}\sqrt{3}\nu_2(9\nu_2 - 13)(\nu_2 + 2\nu_3 - 1) + \frac{\varepsilon}{8}\sqrt{3}\nu_3(9\nu_3 - 13)(2\nu_2 + \nu_3 - 1), \\
\mathcal{N}_{35} &= \frac{3\varepsilon}{32}\sqrt{3}\nu_2[15\nu_2^2 + \nu_2(24\nu_3 - 26) - 3\nu_3^2 - 34\nu_3 + 16] \\
&\quad + \frac{3\varepsilon}{32}\sqrt{3}\nu_3[-3\nu_2^2 + \nu_2(24\nu_3 - 34) + 15\nu_3^2 - 26\nu_3 + 16], \\
\mathcal{N}_{36} &= \frac{3\sqrt{3}\nu_2}{4} - \frac{\varepsilon}{32}\sqrt{3}\nu_2[36\nu_2^2 + \nu_2(21\nu_3 - 4) + 54\nu_3^2 + 53\nu_3 + 103 + \nu_1(3\nu_2 - 6\nu_3 + 5)] \\
&\quad - \frac{3}{4}\sqrt{3}\nu_3 + \frac{3\sqrt{3}\varepsilon}{32}\nu_3[17\nu_2^2 + 4\nu_2(8\nu_3 - 5) + 26\nu_3^2 - 28\nu_3 + 52], \\
\mathcal{N}_{37} &= \frac{9\nu_2}{4} - \frac{3\varepsilon}{32}\nu_2[39\nu_2^2 + \nu_2(39\nu_3 - 64) + 60\nu_3^2 - 37\nu_3 + 150 + \nu_1(3\nu_2 - 6\nu_3 + 5)] \\
&\quad + \frac{9}{4}\nu_3 - \frac{3\varepsilon}{32}\nu_3[66\nu_2^2 + 6\nu_2(7\nu_3 - 8) + 36\nu_3^2 - 66\nu_3 + 155], \\
\mathcal{N}_{41} &= -2 - \frac{\varepsilon}{24}[-6\nu_2^2(9\nu_3 + 17) - 6\nu_2(9\nu_3^2 + 8\nu_3 - 17) + 27\nu_3^3 - 102\nu_3^2 + 105\nu_3 - 193], \\
\mathcal{N}_{42} &= -\frac{\varepsilon}{8}\nu_3(-9\nu_2\nu_3 + \nu_2 + 9\nu_3^2 - 34\nu_3 + 13), \\
\mathcal{N}_{43} &= -\frac{\varepsilon}{8}\sqrt{3}\nu_3[\nu_2(9\nu_3 - 1) + 9\nu_3^2 - 18\nu_3 + 5], \\
\mathcal{N}_{44} &= -\frac{\varepsilon}{8}\sqrt{3}\nu_3(9\nu_3 - 13)(2\nu_2 + \nu_3 - 1), \\
\mathcal{N}_{45} &= -\frac{3\varepsilon}{32}\sqrt{3}\nu_3[-3\nu_2^2 + \nu_2(24\nu_3 - 34) + 15\nu_3^2 - 26\nu_3 + 16], \\
\mathcal{N}_{46} &= \frac{3}{4}\sqrt{3}\nu_3 - \frac{3\sqrt{3}\varepsilon}{32}\nu_3[17\nu_2^2 + 4\nu_2(8\nu_3 - 5) + 26\nu_3^2 - 28\nu_3 + 52], \\
\mathcal{N}_{47} &= -\frac{9}{4}\nu_3 + \frac{3\varepsilon}{32}\nu_3[66\nu_2^2 + 6\nu_2(7\nu_3 - 8) + 36\nu_3^2 - 66\nu_3 + 155],
\end{aligned}$$

$\mathcal{N}_{51} = \mathcal{N}_{62} = \mathcal{N}_{73} = 1$, and the others are 0.

In order to obtain the eigenvalues λ of the matrix \mathcal{N} , let us consider the eigenequation for the matrix \mathcal{N} . This is expressed as

$$\lambda f(\tau) = 0, \quad (44)$$

where

$$f(\tau) = \tau^3 + C\tau^2 + D\tau + E, \quad (45)$$

and we define $\tau = \lambda^2$ and

$$C \equiv \frac{1}{4}[8 - \varepsilon(77 - 10V)], \quad (46)$$

$$D \equiv \frac{1}{16}[4(4 + 27V) + \varepsilon(378V^2 - 1265V - 162W - 308)], \quad (47)$$

$$E \equiv \frac{9}{32}[24V + \varepsilon(126V^2 - 521V + 72W)], \quad (48)$$

with $W \equiv \nu_1\nu_2\nu_3$. For the cubic equation $f(\tau) = 0$, we obtain

$$\begin{aligned} \Delta &= \frac{-C^2D^2 + 4C^3E - 18CDE + 4D^3 + 27E^2}{27} \\ &= \frac{27}{16}(27V - 1)V^2 + \frac{30618V^4 - 105759V^3 + V^2(4657 - 13122W) + 9072VW - 288W}{64}\varepsilon, \end{aligned} \quad (49)$$

where Δ denotes the discriminant of Eq.(45). In general, $f(\tau) = 0$ does not have zero root since $E \neq 0$. On the other hand, if $\tau > 0$, $f(\tau) \neq 0$ because of $\varepsilon \ll 1$. Therefore, $f(\tau) = 0$ does not have positive real roots. In addition, if $f(\tau) = 0$ has roots of complex numbers, the matrix \mathcal{N} has complex eigenvalues which are non-zero real parts. Thus, if $\Delta < 0$, all of the roots of $f(\tau) = 0$ are negative real numbers. These can be expressed as [20]

$$\tau_1 = -1 + a\varepsilon, \quad \tau_{\pm} = \frac{-b \pm \sqrt{b^2 - 4c}}{2}, \quad (50)$$

with

$$a = \frac{1}{8V}(77V - 14V^2 - 36W), \quad (51)$$

$$b = 1 - \frac{1}{8V}(77V - 6V^2 + 36W)\varepsilon, \quad (52)$$

$$c = \frac{27}{4}V - \frac{1}{16}(1305V - 378V^2 + 162W)\varepsilon. \quad (53)$$

One can show that the condition $\Delta < 0$ is equivalent to $b^2 - 4c > 0$ by straightforward calculations. It is the necessary condition for stable system that all of the roots of $f(\tau) = 0$ are negative real numbers. Namely, $b^2 - 4c > 0$ or equivalently at the 1PN order [20]

$$1 - 27V - \left(\frac{391}{54} + \frac{405}{2}W\right)\varepsilon > 0. \quad (54)$$

This is nothing but Eq. (3).

As a result, the eigenvalues of the matrix \mathcal{N} are one zero value and six purely imaginary numbers, namely

$$\lambda_0 = 0, \quad \lambda_{1\pm} = \pm i\sqrt{\frac{b - \sqrt{b^2 - 4c}}{2}}, \quad \lambda_{2\pm} = \pm i\sqrt{\frac{b + \sqrt{b^2 - 4c}}{2}}, \quad \lambda_{3\pm} = \pm i\sqrt{1 - a\varepsilon}. \quad (55)$$

One can find that there is a regular matrix Q , such that $Q^{-1}\mathcal{N}Q$ becomes a diagonalized matrix. Therefore, Eq. (43) is rewritten as

$$D\bar{\chi} = Q^{-1}\mathcal{N}Q\bar{\chi}, \quad (56)$$

$$Q^{-1}\mathcal{N}Q = \text{diag}(0, \lambda_{1+}, \lambda_{1-}, \lambda_{2+}, \lambda_{2-}, \lambda_{3+}, \lambda_{3-}). \quad (57)$$

where $\bar{\chi} = Q^{-1}\chi$. The solution of Eq.(56) can be expressed as

$$\begin{aligned} \bar{\chi} &= \exp(\omega_N t Q^{-1}\mathcal{N}Q) \bar{\chi}_0 \\ &= \text{diag}(1, e^{\omega_N t \lambda_{1+}}, e^{\omega_N t \lambda_{1-}}, e^{\omega_N t \lambda_{2+}}, e^{\omega_N t \lambda_{2-}}, e^{\omega_N t \lambda_{3+}}, e^{\omega_N t \lambda_{3-}}) \bar{\chi}_0, \end{aligned} \quad (58)$$

and equivalently,

$$\chi = Q \text{diag}(1, e^{\omega_N t \lambda_{1+}}, e^{\omega_N t \lambda_{1-}}, e^{\omega_N t \lambda_{2+}}, e^{\omega_N t \lambda_{2-}}, e^{\omega_N t \lambda_{3+}}, e^{\omega_N t \lambda_{3-}}) Q^{-1} \chi_0, \quad (59)$$

where $\bar{\chi}_0 = Q^{-1}\chi_0$ and χ_0 is the initial value. Therefore, we can solve the motion of the perturbations as

$$\begin{cases} \chi_{12} = C_{11} + C_{12}e^{\omega_N t \lambda_{1+}} + C_{13}e^{\omega_N t \lambda_{1-}} + C_{14}e^{\omega_N t \lambda_{2+}} + C_{15}e^{\omega_N t \lambda_{2-}} + C_{16}e^{\omega_N t \lambda_{3+}} + C_{17}e^{\omega_N t \lambda_{3-}}, \\ X = C_{21} + C_{22}e^{\omega_N t \lambda_{1+}} + C_{23}e^{\omega_N t \lambda_{1-}} + C_{24}e^{\omega_N t \lambda_{2+}} + C_{25}e^{\omega_N t \lambda_{2-}} + C_{26}e^{\omega_N t \lambda_{3+}} + C_{27}e^{\omega_N t \lambda_{3-}}, \\ \psi_{23} = C_{31} + C_{32}e^{\omega_N t \lambda_{1+}} + C_{33}e^{\omega_N t \lambda_{1-}} + C_{34}e^{\omega_N t \lambda_{2+}} + C_{35}e^{\omega_N t \lambda_{2-}} + C_{36}e^{\omega_N t \lambda_{3+}} + C_{37}e^{\omega_N t \lambda_{3-}}, \\ \sigma = C_{40} + C_{41}\omega_N t + C_{42}e^{\omega_N t \lambda_{1+}} + C_{43}e^{\omega_N t \lambda_{1-}} + C_{44}e^{\omega_N t \lambda_{2+}} + C_{45}e^{\omega_N t \lambda_{2-}} + C_{36}e^{\omega_N t \lambda_{3+}} \\ \quad + C_{47}e^{\omega_N t \lambda_{3-}}, \end{cases} \quad (60)$$

where C_{ij} is the constant value determined by the initial condition χ_0 . The solutions χ_{12} , X and ψ_{23} give the oscillation around the PN triangular equilibrium since there are only the terms which are the forms as $e^{t\lambda}$ with a purely imaginary number λ . On the other hand, σ includes the term which is linear in time t . This term does not affect the change of the shape of the PN triangle, but gives only the change of the angular velocity of the system regarding the scale transformation. Therefore, the system is stable if and only if the condition (54) is satisfied.

V. CONCLUSION

We reexamined the linear stability of the PN triangular solution for general masses by taking account of perturbations orthogonal to the orbital plane, as well as perturbations lying on it.

We found that while the orthogonal perturbations are independent of the lying ones likewise the Newtonian case, these depend on each other by the 1PN three-body interactions. We also showed that the orthogonal perturbations do not affect the condition of stability. This is because these always precess with two frequency modes; the same with the orbital frequency and the slightly different one, which is caused by the 1PN effect for the first time. The existence of the second frequency mode may occur resonant orbits in nonlinear analysis. This is left as a future work. The same condition of stability, which is valid even for the general perturbations, was obtained from the lying perturbations.

Acknowledgments

We would like to thank Hideki Asada for reading the manuscript. KY is grateful to Takahiro Tanaka, Hiroyuki Nakano, and Hiroyuki Kitamoto for useful comments and encouragements. This work was supported in part by JSPS Grant-in-Aid for JSPS Fellows, No. 15J01732 (K.Y.).

Appendix A: Derivation of EIH equations of motion in uniformly rotating frame

The EIH equation of motion for N -body systems is give by [37–40]

$$\begin{aligned} \frac{d^2 \mathbf{r}_K}{dt^2} = & \sum_{A \neq K} \frac{Gm_A}{r_{AK}^3} \mathbf{r}_{AK} \left[1 - 4 \sum_{B \neq K} \frac{Gm_B}{c^2 r_{BK}} - \sum_{C \neq A} \frac{Gm_C}{c^2 r_{CA}} \left(1 - \frac{\mathbf{r}_{AK} \cdot \mathbf{r}_{CA}}{2r_{CA}^2} \right) + \frac{v_K^2}{c^2} + 2 \frac{v_A^2}{c^2} \right. \\ & \left. - 4 \frac{\mathbf{v}_A \cdot \mathbf{v}_K}{c^2} - \frac{3}{2} \left(\frac{\mathbf{v}_A}{c} \cdot \mathbf{x}_{AK} \right)^2 \right] - \sum_{A \neq K} \frac{Gm_A}{c^2 r_{AK}^2} \mathbf{x}_{AK} \cdot \left(3 \frac{\mathbf{v}_A}{c} - 4 \frac{\mathbf{v}_K}{c} \right) \left(\frac{\mathbf{v}_A}{c} - \frac{\mathbf{v}_K}{c} \right) \\ & + \frac{7}{2} \sum_{A \neq K} \sum_{C \neq A} \frac{G^2 m_A m_C}{c^2 r_{AK} r_{CA}^3} \mathbf{r}_{CA}, \end{aligned} \quad (\text{A1})$$

where \mathbf{r}_I , \mathbf{v}_I , m_I are the position, the velocity, and the mass of I -th particle, respectively. $\mathbf{r}_{IJ} \equiv \mathbf{r}_I - \mathbf{r}_J$, $r_{IJ} \equiv |\mathbf{r}_{IJ}|$, $\mathbf{x}_{IJ} \equiv \mathbf{r}_{IJ}/r_{IJ}$, $v_I \equiv |\mathbf{v}_I|$. For the above equation, we consider the linear transformation of the function t . In general, a linear transformation from \mathbb{R}^n to \mathbb{R}^n is given by

$$\mathbf{r}' = \mathbf{R} \mathbf{r}, \quad (\text{A2})$$

where $\mathbf{r}, \mathbf{r}' \in \mathbb{R}^n$, \mathbf{R} is an $n \times n$ matrix. If \mathbf{R} is a one to one and onto mappings, the linear mapping of the first order and second order time derivatives of \mathbf{r} are calculated as

$$\mathbf{R} \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}'}{dt} + \mathbf{S} \mathbf{r}', \quad (\text{A3})$$

$$\mathbf{R} \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2\mathbf{r}'}{dt^2} + 2\mathbf{S} \frac{d\mathbf{r}'}{dt} + \mathbf{S}^2 \mathbf{r}' + \frac{d\mathbf{S}}{dt} \mathbf{r}', \quad (\text{A4})$$

where $\mathbf{S} \equiv -(\frac{d\mathbf{R}}{dt})\mathbf{R}^{-1}$.

If \mathbf{R} is a rotation matrix, \mathbf{R} is the orthogonal matrix and the determinant of \mathbf{R} is 1. Therefore, the transposition of \mathbf{R} is consistent with the inverse of \mathbf{R} , namely ${}^t\mathbf{R} = \mathbf{R}^{-1}$, and \mathbf{S} becomes skew-symmetric. Thus, for three dimensional transformation, we can set \mathbf{S} as

$$\mathbf{S} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}, \quad (\text{A5})$$

then for all vector $\mathbf{v} \in \mathbb{R}^3$, it is satisfied the relation such as $\mathbf{S}\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{v}$, where $\boldsymbol{\Omega} = (w_1, w_2, w_3)$.

The EIH equation in a uniformly rotating frame of constant angular velocity $\boldsymbol{\Omega}$ is can be expressed as

$$\begin{aligned} \frac{d^2\mathbf{r}'_K}{dt^2} = & \sum_{A \neq K} \frac{Gm_A}{(r'_{KA})^3} \mathbf{r}'_{AK} - 2(\boldsymbol{\Omega} \times \mathbf{v}'_K) - (\boldsymbol{\Omega} \cdot \mathbf{r}'_K) \boldsymbol{\Omega} + \boldsymbol{\Omega}^2 \mathbf{r}'_K \\ & + \sum_{A \neq K} \frac{Gm_A}{(r'_{KA})^3} \mathbf{r}'_{AK} \left[-4 \sum_{B \neq K} \frac{Gm_B}{c^2 r'_{KB}} - \sum_{C \neq A} \frac{Gm_C}{c^2 r'_{AC}} \left(1 + \frac{\mathbf{r}'_{AK} \cdot \mathbf{r}'_{AC}}{2(r'_{CA})^2} \right) \right. \\ & + \left(\frac{\mathbf{v}'_K + (\boldsymbol{\Omega} \times \mathbf{r}'_K)}{c} \right)^2 + 2 \left(\frac{\mathbf{v}'_A + (\boldsymbol{\Omega} \times \mathbf{r}'_A)}{c} \right)^2 \\ & - 4 \left(\frac{\mathbf{v}'_K + (\boldsymbol{\Omega} \times \mathbf{r}'_K)}{c} \right) \cdot \left(\frac{\mathbf{v}'_A + (\boldsymbol{\Omega} \times \mathbf{r}'_A)}{c} \right) - \frac{3}{2} \left\{ \left(\frac{\mathbf{v}'_A + (\boldsymbol{\Omega} \times \mathbf{r}'_A)}{c} \right) \cdot \mathbf{x}'_{AK} \right\}^2 \Big] \\ & - \sum_{A \neq K} \frac{Gm_A}{c^2 (r'_{KA})^2} \left[\mathbf{x}'_{AK} \cdot \left(\frac{4[\mathbf{v}'_K + (\boldsymbol{\Omega} \times \mathbf{r}'_K)] - 3[\mathbf{v}'_A + (\boldsymbol{\Omega} \times \mathbf{r}'_A)]}{c} \right) \right] \\ & \times \left(\frac{[\mathbf{v}'_K + (\boldsymbol{\Omega} \times \mathbf{r}'_K)] - [\mathbf{v}'_A + (\boldsymbol{\Omega} \times \mathbf{r}'_A)]}{c} \right) \\ & + \frac{7}{2} \sum_{A \neq K} \sum_{C \neq A} \frac{Gm_A}{c^2 r'_{KA}} \frac{Gm_C}{(r'_{AC})^3} \mathbf{r}'_{CA}, \end{aligned} \quad (\text{A6})$$

with using the relations such as

$$\mathbf{r}_{IJ} \cdot \mathbf{r}_{MN} = \mathbf{r}'_{IJ} \cdot \mathbf{r}'_{MN}, \quad (\text{A7})$$

$$\mathbf{v}_I \cdot \mathbf{v}_J = (\mathbf{v}'_I + \boldsymbol{\Omega} \times \mathbf{r}'_I) \cdot (\mathbf{v}'_J + \boldsymbol{\Omega} \times \mathbf{r}'_J), \quad (\text{A8})$$

$$\mathbf{v}_I \cdot \mathbf{r}_{JK} = (\mathbf{r}'_I + \boldsymbol{\Omega} \times \mathbf{r}'_I) \cdot \mathbf{r}'_{JK}, \quad (\text{A9})$$

and we set $\Omega \equiv |\boldsymbol{\Omega}|$

Appendix B: Transformation to Routh's variables

Eliminating ξ_{23} and η_{23} from Eq. (25), the perturbed equations of motion for ξ_{12} , η_{12} , ξ_{31} , η_{31} become

$$\begin{aligned} 0 = & \left(D^2 + \frac{9\nu_3}{4} - 3 \right) \xi_{12} + \left(\frac{3\sqrt{3}\nu_3}{4} - 2D \right) \eta_{12} - \frac{3}{4}\sqrt{3}\nu_3\eta_{31} - \frac{9}{4}\nu_3\xi_{31} \\ & + \varepsilon \left[\left(\frac{1}{8}\sqrt{3}D\nu_3(-9\nu_2\nu_3 + 21\nu_2 + 4\nu_3 - 8) \right. \right. \\ & + \frac{1}{32}[-22\nu_2^2(9\nu_3 + 4) + \nu_2(-126\nu_3^2 + 136\nu_3 + 88) - 108\nu_3^3 + 270\nu_3^2 - 537\nu_3 + 540] \Big) \xi_{12} \\ & + \left(\frac{1}{8}D[-2\nu_2^2(9\nu_3 + 17) + \nu_2(-9\nu_3^2 - 9\nu_3 + 34) + 10\nu_3^2 + 2\nu_3 + 45] \right. \\ & - \frac{1}{32}\sqrt{3}\nu_3[30\nu_2^2 + \nu_2(66\nu_3 + 56) + 84\nu_3^2 - 66\nu_3 + 127] \Big) \eta_{12} \\ & + \left(\frac{1}{32}\nu_3[126\nu_2^2 + 2\nu_2(27\nu_3 - 76) + 144\nu_3^2 - 322\nu_3 + 553] \right. \\ & - \frac{1}{8}\sqrt{3}D\nu_3[\nu_2(9\nu_3 + 7) + 9\nu_3^2 - 12\nu_3 - 1] \Big) \xi_{31} \\ & + \left(\frac{1}{8}D\nu_3[-\nu_2(9\nu_3 + 7) + 9\nu_3^2 - 32\nu_3 + 11] \right. \\ & \left. \left. + \frac{1}{32}\sqrt{3}\nu_3[30\nu_2^2 + \nu_2(66\nu_3 + 56) + 84\nu_3^2 - 66\nu_3 + 127] \right) \eta_{31} \right], \quad (\text{B1}) \end{aligned}$$

$$\begin{aligned}
0 = & \left(D^2 - \frac{9\nu_3}{4} \right) \eta_{12} + \left(2D + \frac{3\sqrt{3}\nu_3}{4} \right) \xi_{12} + \frac{9}{4}\nu_3\eta_{31} - \frac{3}{4}\sqrt{3}\nu_3\xi_{31} \\
& + \varepsilon \left[\left(\frac{1}{8}D \left[-6\nu_2^2(3\nu_3 + 5) + \nu_2(-9\nu_3^2 - 13\nu_3 + 30) - 2\nu_3^2 + 14\nu_3 - 61 \right] \right. \right. \\
& \left. \left. - \frac{1}{32}\sqrt{3}\nu_3 \left[54\nu_2^2 + 6\nu_2(3\nu_3 + 8) + 36\nu_3^2 + 6\nu_3 + 103 \right] \right) \xi_{12} \right. \\
& + \left(\frac{1}{8}\sqrt{3}D\nu_3 \left[\nu_2(9\nu_3 - 25) - 4\nu_3 + 8 \right] + \frac{3}{32}\nu_3 \left[66\nu_2^2 + 6\nu_2(7\nu_3 - 8) + 36\nu_3^2 - 66\nu_3 + 155 \right] \right) \eta_{12} \\
& + \left(\frac{1}{8}D\nu_3 \left(-9\nu_2\nu_3 + \nu_2 + 9\nu_3^2 - 34\nu_3 + 13 \right) \right. \\
& + \left. \frac{1}{32}\sqrt{3}\nu_3 \left[54\nu_2^2 + 6\nu_2(15\nu_3 - 8) + 72\nu_3^2 - 78\nu_3 + 151 \right] \right) \xi_{31} \\
& + \left(\frac{1}{8}\sqrt{3}D\nu_3 \left[\nu_2(9\nu_3 - 1) + 9\nu_3^2 - 18\nu_3 + 5 \right] \right. \\
& \left. \left. - \frac{3}{32}\nu_3 \left[66\nu_2^2 + 6\nu_2(7\nu_3 - 8) + 36\nu_3^2 - 66\nu_3 + 155 \right] \right) \eta_{31} \right], \tag{B2}
\end{aligned}$$

$$\begin{aligned}
0 = & \left(D^2 + \frac{9\nu_2}{4} - 3 \right) \xi_{31} + \left(-2D - \frac{3\sqrt{3}\nu_2}{4} \right) \eta_{31} + \frac{3}{4}\sqrt{3}\nu_2\eta_{12} - \frac{9}{4}\nu_2\xi_{12} \\
& + \varepsilon \left[\left(\frac{1}{8}\sqrt{3}D\nu_2 \left[9\nu_2^2 + 3\nu_2(3\nu_3 - 4) + 7\nu_3 - 1 \right] \right. \right. \\
& + \frac{1}{32}\nu_2 \left[144\nu_2^2 + \nu_2(54\nu_3 - 322) + 126\nu_3^2 - 152\nu_3 + 553 \right] \right) \xi_{12} \\
& + \left(\frac{1}{8}D\nu_2 \left[9\nu_2^2 - \nu_2(9\nu_3 + 32) - 7\nu_3 + 11 \right] \right. \\
& \left. - \frac{1}{32}\sqrt{3}\nu_2 \left[84\nu_2^2 + 66\nu_2(\nu_3 - 1) + 30\nu_3^2 + 56\nu_3 + 127 \right] \right) \eta_{12} \\
& + \left(\frac{1}{8}\sqrt{3}D\nu_2 \left[\nu_2(9\nu_3 - 4) - 21\nu_3 + 8 \right] \right. \\
& + \frac{1}{32} \left[-108\nu_2^3 - 18\nu_2^2(7\nu_3 - 15) + \nu_2(-198\nu_3^2 + 136\nu_3 - 537) - 88\nu_3^2 + 88\nu_3 + 540 \right] \right) \xi_{31} \\
& + \left(\frac{1}{8}D \left[\nu_2^2(10 - 9\nu_3) + \nu_2(-18\nu_3^2 - 9\nu_3 + 2) - 34\nu_3^2 + 34\nu_3 + 45 \right] \right. \\
& \left. + \frac{1}{32}\sqrt{3}\nu_2 \left[84\nu_2^2 + 66\nu_2(\nu_3 - 1) + 30\nu_3^2 + 56\nu_3 + 127 \right] \right) \eta_{31} \right], \tag{B3}
\end{aligned}$$

$$\begin{aligned}
0 = & \left(D^2 - \frac{9\nu_2}{4} \right) \eta_{31} + \left(2D - \frac{3\sqrt{3}\nu_2}{4} \right) \xi_{31} + \frac{9}{4}\nu_2\eta_{12} + \frac{3}{4}\sqrt{3}\nu_2\xi_{12} \\
& + \varepsilon \left[\left(\frac{1}{8}D\nu_2 [9\nu_2^2 - \nu_2(9\nu_3 + 34) + \nu_3 + 13] \right. \right. \\
& \left. \left. - \frac{1}{32}\sqrt{3}\nu_2 [72\nu_2^2 + \nu_2(90\nu_3 - 78) + 54\nu_3^2 - 48\nu_3 + 151] \right) \xi_{12} \right. \\
& + \left(\frac{1}{8}\sqrt{3}D\nu_2 [-9\nu_2^2 - 9\nu_2(\nu_3 - 2) + \nu_3 - 5] \right. \\
& \left. - \frac{3}{32}\nu_2 [36\nu_2^2 + 6\nu_2(7\nu_3 - 11) + 66\nu_3^2 - 48\nu_3 + 155] \right) \eta_{12} \\
& + \left(\frac{1}{8}D [\nu_2^2(-(9\nu_3 + 2)) + \nu_2(-18\nu_3^2 - 13\nu_3 + 14) - 30\nu_3^2 + 30\nu_3 - 61] \right. \\
& \left. + \frac{1}{32}\sqrt{3}\nu_2 [36\nu_2^2 + 6\nu_2(3\nu_3 + 1) + 54\nu_3^2 + 48\nu_3 + 103] \right) \xi_{31} \\
& \left. + \left(\frac{1}{8}\sqrt{3}D\nu_2[\nu_2(4 - 9\nu_3) + 25\nu_3 - 8] + \frac{3}{32}\nu_2 [36\nu_2^2 + 6\nu_2(7\nu_3 - 11) + 66\nu_3^2 - 48\nu_3 + 155] \right) \eta_{31} \right].
\end{aligned} \tag{B4}$$

Let us seek the relations between (ξ_{IJ}, η_{IJ}) and Routh's variables. First, since χ_{12} is a perturbations to r_{12} , we obtain the relation as

$$\ell(1 + \rho_{12})(1 + \xi_{12}) = \ell(1 + \rho_{12} + \chi_{12}). \tag{B5}$$

Therefore,

$$\chi_{12} = (1 + \rho_{12})\xi_{12}. \tag{B6}$$

In the same way, we obtain the relation for X as

$$X = (1 + \rho_{31})\xi_{31} - (1 + \rho_{12})\xi_{12}. \tag{B7}$$

Next, let us define the projection of vectors onto the orbital plane as

$$\bar{\mathbf{A}} \equiv \mathbf{A} - (\mathbf{A} \cdot \mathbf{z})\mathbf{z}. \tag{B8}$$

Using this, σ is expressed as a perturbation to the angle between $\bar{\mathbf{r}}_{12}$ and $\bar{\mathbf{r}}_{12} + \delta\bar{\mathbf{r}}_{12}$, and then,

$$\sin \sigma = \frac{|\bar{\mathbf{r}}_{12} \times (\bar{\mathbf{r}}_{12} + \delta\bar{\mathbf{r}}_{12})|}{r_{12}^2(1 + \xi_{12})}. \tag{B9}$$

Solving this for σ to the 1PN order, we obtain

$$\sigma = \eta_{12}. \quad (\text{B10})$$

Finally, since ψ_{23} is a perturbation in the opposite angle of r_{23} , we obtain

$$\cos\left(\frac{\pi}{3} + \sqrt{3}\rho_{23} + \psi_{23}\right) = -\frac{(\bar{\mathbf{r}}_{31} + \delta\bar{\mathbf{r}}_{31}) \cdot (\bar{\mathbf{r}}_{12} + \delta\bar{\mathbf{r}}_{12})}{r_{31}r_{12}(1 + \xi_{31})(1 + \xi_{12})}. \quad (\text{B11})$$

This leads to

$$\psi_{23} = \eta_{31} - \eta_{12}, \quad (\text{B12})$$

at the 1PN order. Using these relations, we obtain the perturbed equations of motion (39)-(42).

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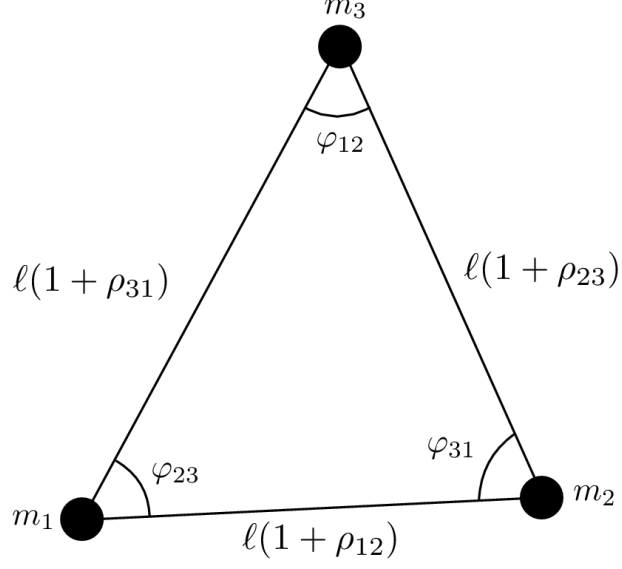


FIG. 1: PN triangular configuration. Each body is located at one of the apexes. ρ_{IJ} denotes the PN corrections to each side length at the 1PN order. In the equilateral case, $\rho_{12} = \rho_{23} = \rho_{31} = 0$, namely, $r_{12} = r_{23} = r_{31} = \ell$ according to Eq. (11).

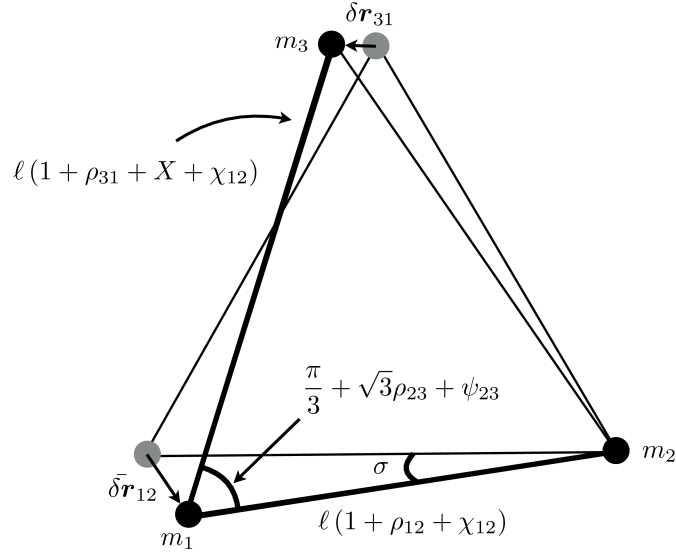


FIG. 2: Perturbations in the orbital plane with Routh's variables, in which we consider the relative perturbations to \mathbf{r}_1 and \mathbf{r}_3 with fixed \mathbf{r}_2 . $\bar{\delta}\mathbf{r}_{12}$ and $\bar{\delta}\mathbf{r}_{31}$ are the projected vector onto the plane. χ_{12} and σ correspond to the scale transformation of the triangle and the change of the angle of the system to a reference direction, respectively. On the other hand, X and ψ_{23} are the degrees of freedom of a shape change from the equilateral triangle.